# Equilibrium Problems and Riesz Representation Theorem

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# Equilibrium Problems and Riesz Representation Theorem

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**Abstract**

The purpose of this paper is to give an alternative proof of the Riesz representation theorem using the well-known theorem of Ky Fan minimax inequality applied to equilibrium problems.

**Key words:** Equilibrium problem, Convexity, Monotonicity, Coercivity.

## 1 Introduction

Let $H$ be a Hilbert space, and let $H'$ denote its dual space, consisting of all continuous linear functionals from $H$ into the field $\mathbb{R}$. It is very well known that for each $x \in H$, the function $x^* : H \to \mathbb{R}$ defined by

$$x^*(y) = \langle x, y \rangle, \text{ for all } y \in H$$

where $\langle \cdot, \cdot \rangle$ denote the inner product of $H$, is an element to $H'$. The Riesz Representation Theorem states that every element of $H'$ can be written uniquely in this form (see for instance [2]).

On the other hand, Blum and Oettli introduced the *equilibrium problems* in 1993 (see [1]), as a generalization of various problems such as minimization problems, Nash equilibrium, variational inequalities, etc (see for instance [1, 3]).

Formally, an equilibrium problem, associated to $K \subset H$ and $f : K \times K \to \mathbb{R}$, consists on finding $x \in K$ such that

$$f(x, y) \geqslant 0, \text{ for all } y \in K.$$

The solution set of an equilibrium problem is denoted by $EP(f, K)$.

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## 2 Preliminaries

Let $H$ be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$. Let $K$ be a convex subset of $H$. Recall that a function $h : K \to \mathbb{R}$ is said to be:

- *convex* when for all $u, v \in K$ and all $t \in ]0, 1[$ the following holds

$$h(tu + (1-t)v) \leqslant th(u) + (1-t)h(v)$$

- *quasiconvex* when for all $u, v \in K$ and all $t \in ]0, 1[$ the following holds

$$h(tu + (1-t)v) \leqslant \max\{h(u), h(v)\}$$

- *lower semicontinuous* when for each $x_0 \in K$ and each $\lambda < f(x_0)$ there exists $\delta > 0$ such that for all $x \in K$ the following implication holds

$$\|x - x_0\| < \delta \ \Rightarrow \ f(x) > \lambda.$$

Finally, a function $f$ is called *concave*, *quasiconcave* or *upper semicontinuous* if $-f$ is convex, quasiconvex or lower semicontinuous, respectively.

Clearly, all convex functions are quasiconvex. Additionally, a function is continuous if, and only if, it is lower and upper semicontinuous.

A function $f : K \times K \to \mathbb{R}$, defined on $K \subset H$, is called:

- *monotone* when $f(u, v) + f(v, u) \leqslant 0$, for all $u, v \in K$;
- *coercive* when for all sequence $(u_m) \subset K$ with $\|u_m\| \to +\infty$ there exists $u \in K$ such that $f(u_n, u) \leqslant 0$, for all $n$ large enough.

Usually, the function $f$ is called *bifunction*.

The following result is due to Ky Fan who proved the famous *minimax inequality*.

**Theorem 2.1** ([4], Theorem 1)**.** *Let $V$ be a real Hausdorff topological vector space and $K$ a nonempty compact convex subset of $V$. If a bifunction $f : K \times K \to \mathbb{R}$ satisfies:*

- *$f(\cdot, y) : K \to \mathbb{R}$ is upper semicontinuous for each $y \in K$,*
- *$f(x, \cdot) : K \to \mathbb{R}$ is quasiconvex for each $x \in K$,*

*then there exists a point $x \in K$ such that*

$$\inf_{y \in K} f(x, y) \geqslant \inf_{w \in K} f(w, w).$$

The above theorem plays an important rol in equilibrium problems, because Ky Fan's Theorem implies existence of solutions for these problems.

## 3 An altertive proof of Riesz Representation Theorem

For each $\phi \in H'$, we define the bifunction $f : H \times H \to \mathbb{R}$ as

$$f(u, v) = \phi(u - v) - \langle u, u - v\rangle \text{ for all } u, v \in H \tag{R}$$

We note that $f$ satisfies the following property:

$(i)$ $f(u, u) = 0$, for all $u \in H$.

$(ii)$ Since $\phi$ and $\langle\cdot,\cdot\rangle$ are both continuous, we have that $f$ is continuous.

$(iii)$ For all $u, v \in H$,

$$f(u,v)+f(v,u) = \phi(u-v)-\langle u, u-v\rangle+\phi(v-u)-\langle v, v-u\rangle = -\|u-v\|^2 \leqslant 0$$

thus, $f$ is monotone.

$(iv)$ Let $u \in H$ we note that

$$f(u, \cdot) = \phi(u - \cdot) - \langle u, u - \cdot\rangle.$$

therefore, the function $f(u, \cdot)$ is affine.

$(v)$ For each $v \in H$,

$$f(\cdot, v) = \phi(\cdot - v) - \|\cdot\|^2 + \langle\cdot, v\rangle$$

The linearity of $\phi$ and $\langle\cdot, v\rangle$, and concavity of $-\|\cdot\|^2$ imply that $f(\cdot, v)$ is concave.

**Theorem 3.1** (Riesz Representation Theorem)**.** *For each $\phi \in H'$, there exists an unique $u_0 \in H$ such that*

$$\phi(u) = \langle u_0, u\rangle \text{ for all } u \in H.$$

*Moreover,* $\|u_0\| = \|\phi\|_{H'}$.

In order to proof the above proposition we need the following lemmas.

**Lemma 3.2.** *Let $f$ defined as (R). If $u_1, u_2 \in EP(f, H)$ then $u_1 = u_2$.*

*Proof.* The monotonicity of $f$ implies $f(u_1, u_2) = f(u_2, u_1) = 0$. So,

$$0 = f(u_1, u_2) + f(u_2, u_1) = \|u_1 - u_2\|^2$$

Therefore $u_1 = u_2$. □

*Proof of Theorem 3.1.* The uniqueness follows from Lemma 3.2.

For the existence, we consider $K = \overline{B}(0, \|\phi\|_{H'})$ which is weakly compact and convex. Since the bifunction (R) satisfies the conditions of Theorem 2.1 on $K$, there exists $u_0 \in K$ such that

$$\inf_{v \in K} f(u_0, v) \geqslant \inf_{w \in K} f(w, w) = 0$$

i.e. $f(u_0, v) \geqslant 0$ for all $v \in K$ and in particular $f(u_0, 0) = \phi(u_0) - \|u_0\|^2 \geqslant 0$. We want to show that $f(u_0, \cdot)$ is linear. Since $f(u, \cdot)$ is affine it is enough to show that $f(u_0, 0) = \phi(u) - \|u\|^2 = 0$. Suppose that $\|u_0\|^2 < \phi(u_0)$, then

$$0 \leqslant \|u_0\|^2 < \phi(u_0) = |\phi(u_0)| \leqslant \|\phi\|_{H'} \|u_0\| \quad \rightarrow \quad \|u_0\| < \|\phi\|_{H'}$$

and so there exists $t > 1$ such that $tu_0 \in K$. Thus,

$$f(u_0, tu_0) = (1-t)\phi(u_0) - (1-t)\|u_0\|^2 = (1-t)[\phi(u_0) - \|u_0\|^2] \geqslant 0$$

and this implies $\phi(u_0) - \|u_0\|^2 \leqslant 0$ which is a contradiction. Therefore, we have $\phi(u_0) = \|u_0\|^2$ and $f(u_0, \cdot)$ is linear.

So, for each $v \notin K$ there exists $t \in ]0,1[$ such that $tv \in K$. The linearity of $f(u_0, \cdot)$ implies $f(u_0, v) = t^{-1} \times f(u_0, tv) \geqslant 0$. Thus, $u_0 \in EP(f, H)$. Also by linearity of $f(u_0, \cdot)$ we have $f(u_0, -v) \geqslant 0$ which is true if and only if $f(u_0, v) \leqslant 0$ and therefore

$$f(u_0, v) = \phi(u_0 - v) - \langle u_0, u_0 - v \rangle = 0, \text{ for all } v \in H$$

which is equivalent to $\phi(w) = \langle u_0, w \rangle$, for all $w \in H$.

Finally, we note that

$$\|\phi\|_{H'} = \sup_{u \in H,\ \|u\| \leqslant 1} \phi(u) = \sup_{u \in H,\ \|u\| \leqslant 1} \langle u_0, u \rangle \leqslant \|u_0\| \leqslant \|\phi\|_{H'}$$

therefore $\|\phi\|_{H'} = \|u_0\|$.

$\square$

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